

KILLING WILD RAMIFICATION

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ABSTRACT. We compute the inertia group of the compositum of wildly ramified Galois covers. It is used to show that even the p -part of the inertia group of a Galois cover of \mathbb{P}^1 branched only at infinity can be reduced if there is a jump in the ramification filtration at two (in the lower numbering) and certain linear disjointness statement holds.

1. INTRODUCTION

Let k be a field of characteristic p . Let $\phi : X \rightarrow Y$ be a finite Galois G -cover of regular irreducible k -curves branched at $\tau \in Y$. Let I be the inertia subgroup of G at a point of X above τ . It is well known, $I = P \rtimes \mu_n$ where P is a p -group, μ_n is a cyclic group of order n and $(n, p) = 1$. Abhyankar's lemma can be viewed as a tool to modify the tame part of the inertia group. For instance, suppose k contains n^{th} -roots of unity. Let y be a regular local parameter of Y at τ . Let $Z \rightarrow Y$ be the Kummer cover of regular curves given by the field extension $k(Y)[y^{1/n}]/k(Y)$ and $\tau' \in Z$ be the unique point lying above τ . Then the pullback of the cover $X \rightarrow Y$ to Z is a Galois cover of Z branched at τ' . But the inertia group at any point above τ' is P . A wild analogue of this phenomenon appears as Theorem 3.5.

Assume k is also algebraically closed field and let $X \rightarrow \mathbb{P}^1$ be a Galois G -cover of k -curves branched only at ∞ . Let I be the inertia subgroup at some point above ∞ and P be the sylow- p subgroup of I . Then noting that the tame fundamental group of \mathbb{A}^1 is trivial, it can be seen that the conjugates of P in G generate the whole of G . Abhyankar's inertia conjecture states that the converse should also be true. More precisely, any subgroup of a quasi- p group G of the form $P \rtimes \mu_n$ where P is a p -group and $(n, p) = 1$ such that conjugates of P generate G is the inertia group of a G -cover of \mathbb{P}^1 branched only at ∞ .

An immediate consequence of a result of Harbater ([Ha1, Theorem 2]) shows that the inertia conjecture is true for every sylow- p subgroup of G . In fact Harbater's result shows that if a p -subgroup P of G occurs as the inertia group of a G -cover of \mathbb{P}^1 branched only at ∞ and Q is a p -subgroup of G containing P then there exists a G -cover of \mathbb{P}^1 branched only at ∞ so that the inertia group is Q . Proposition 3.4 and a study of wild ramification filtration (Proposition 2.6) enables us to show that in certain cases the given G -cover of \mathbb{P}^1 can be modified to obtain a G -cover of \mathbb{P}^1 branched only at ∞ so that the inertia group of this new cover is smaller than the inertia group P of the original cover (Theorem 3.6).

So far the inertia conjecture is only known for some explicit groups. See for instance [BP, Theorem 5] and [MP, Theorem 1.1].

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2. FILTRATION ON RAMIFICATION GROUP

For a complete discrete valuation ring (DVR) R , v_R will denote the valuation associated to R with the value group \mathbb{Z} . Let S/R be a finite extension of complete DVRs such that $\text{QF}(S)/\text{QF}(R)$ is a Galois extension with Galois group G . Let us define a decreasing filtration on G by

$$G_i = \{\sigma \in G : v_S(\sigma x - x) \geq i + 1, \forall x \in S\}$$

Note that $G_{-1} = G$ and G_0 is the inertia subgroup. This filtration is called the ramification filtration. For every i , G_i is a normal subgroup of G . The following are some well-known results.

Proposition 2.1. [Ser, IV, 1, Proposition 2 and 3] *Let S/R be a finite extension of complete DVRs such that $\text{Gal}(\text{QF}(S)/\text{QF}(R)) = G$. Let H be a subgroup G . Let K be the fixed subfield of $\text{QF}(S)$ under the action of H . Let T be the normalization of R in K . Then T is a complete DVR, $\text{Gal}(\text{QF}(S)/K) = H$ and the ramification filtration on H is induced from that of G , i.e. $H_i = G_i \cap H$. Moreover, if $H = G_j$ for some $j \geq 0$ then $(G/H)_i = G_i/H$ for $i \leq j$ and $(G/H)_i = \{e\}$ for $i \geq j$.*

Proposition 2.2. [Ser, IV, 2, Corollary 2 and 3] *The quotient group G_0/G_1 is a prime-to- p cyclic group and if the residue field has characteristic $p > 0$ then for $i \geq 1$, G_i/G_{i+1} is an elementary abelian group of exponent p . In particular G_1 is a p -group.*

Lemma 2.3. *Let S/R be an extension of DVRs such that $\text{QF}(S)/\text{QF}(R)$ is Galois with $\text{Gal}(\text{QF}(S)/\text{QF}(R)) = G$. Let H be a normal subgroup of G and T be the normalization of R in $\text{QF}(S)^H$ then*

$$\sum_{i=0}^{\infty} (|G_i| - 1) = e_{S/T} \sum_{i=0}^{\infty} (|(G/H)_i| - 1) + \sum_{i=0}^{\infty} (|H_i| - 1)$$

Proof. This follows from the transitivity of the different $\mathcal{D}_{S/R} = \mathcal{D}_{S/T} \mathcal{D}_{T/R}$ [Ser, III, 4, Proposition 8], Hilbert's different formula $d_{S/R} = \sum_{i=0}^{\infty} (|G_i| - 1)$ ([Sti, Theorem 3.8.7]) and $v_S(x) = e_{S/T} v_T(x)$ for $x \in \text{QF}(T)$. \square

Lemma 2.4. *Let S/R be a totally ramified extension of complete DVRs over a perfect field k of characteristic $p > 0$. Suppose $\text{QF}(S)$ is generated over $\text{QF}(R)$ by $\alpha \in \text{QF}(S)$ with $\alpha^p - \alpha \in \text{QF}(R)$ and $v_R(\alpha^p - \alpha) = -1$. Then the degree of the different $d_{S/R} = 2|G| - 2$.*

Proof. Note that since S/R is totally ramified, their residue fields are same and by [Coh] the residue field is isomorphic to the field of coefficient of R and S . Replacing k by this residue field we may assume that the residue fields of S and R are k .

We know that $|G| = p^l$ for some $l \geq 0$. We will prove the lemma by induction on l . If $l = 0$ then the statement is trivial. Suppose $l = 1$. Then by hypothesis there exists $\alpha \in \text{QF}(S)$ with $\alpha^p - \alpha \in R$ and $v_R(\alpha^p - \alpha) = -1$. Let $x = (\alpha^p - \alpha)^{-1}$ and $y = \alpha^{-1}$ then $v_S(x) = e_{S/R} v_R(x) = p$ and $v_S(y) = 1$. By Cohen structure theorem $R = k[[x]]$ and $S = k[[y]]$. Also we have that $m(y) = 0$ where $m(T) = T^p + xT^{p-1} - x \in R[[T]]$. So $m(T)$ is a minimal polynomial of y over $\text{QF}(R)$. By [Ser, III, 6, Corollary 2], $d_{S/R} = v_S(m'(y))$. But $m'(y) = -xy^{p-2}$. So $d_{S/R} = v_S(x) + (p-2)v_S(y) = 2p-2$.

Now in general assume $l \geq 1$. Note that $G = (\mathbb{Z}/p\mathbb{Z})^l$, so by hypothesis there exist $\alpha_1, \dots, \alpha_l \in \text{QF}(S)$ such that

- (1) $\alpha_i^p - \alpha_i = u_i x^{-1}$ for some units $u_i \in R$ and
- (2) $\text{QF}(R)(\alpha_i)$ and $L_{i-1} = \text{QF}(R)(\alpha_j | 1 \leq j < i)$ are linearly disjoint over $\text{QF}(R)$ for $1 \leq i \leq l$.

Note that $L_0 = \text{QF}(R)$ and $L_l = \text{QF}(S)$. Let T_i be the normalization of R in L_i for $0 \leq i \leq l$. Let $y_0 = x$. For simplicity, let v_i denote the valuation v_{T_i} for $0 \leq i \leq l$. Since S/R is totally ramified, so is T_i/T_{i-1} for $1 \leq i \leq l$. Hence $e_{T_i/T_{i-1}} = p$. Note that $v_0(y_0) = 1$.

Claim. For each $0 \leq i \leq l-1$ and $i < j \leq l$, there exist $\beta_{i,j} \in \text{QF}(S)$ such that the following holds

- (1) $\beta_{i,j}^p - \beta_{i,j} \in L_i$,
- (2) $v_i(\beta_{i,j}^p - \beta_{i,j}) = -1$,
- (3) $L_i(\beta_{i,j}; i < j \leq n) = L_n$ for $i < n \leq l-1$
- (4) $v_{i+1}(\beta_{i,i+1}) = -1$

We define $y_{i+1} = \beta_{i,i+1}^{-1}$.

Proof of the claim. We shall proof this by induction. For $i = 0$, we take $\beta_{0,j} = \alpha_j$. The first and the second statement is same as the hypothesis of the lemma. The third statement follows from the definition of L_n 's. For the fourth statement note that $\beta_{0,1} = \alpha_1$. Since $v_1(\alpha_1) < 0$, we have $v_1(\alpha_1^p) = v_1(\alpha_1^p - \alpha_1) = v_1(x^{-1})$. So $v_1(\alpha_1) = p^{-1}v_1(x^{-1}) = p^{-1}pv_0(x^{-1}) = -1$.

Suppose the claim is true for a fixed $i \geq 0$ and $i < l-1$. Then we have $\beta_{i,j} \in \text{QF}(S)$ for $i < j \leq l$ satisfying the four properties listed in the claim. Also note that $v_i(y_i) = 1$. So $T_i = k[[y_i]]$. Hence we can write explicitly $\beta_{i,j}^p - \beta_{i,j} = c_j y_i^{-1} + d_j + f_j(y_i)$ where $c_j, d_j \in k$, $c_j \neq 0$ and $f_j(y_i) \in T_i$ has order at least 1. Let $g_j = f_j + f_j^p + f_j^{p^2} + \dots \in T_i$ then $g_j - g_j^p = f_j$. Let $\gamma_{i,j} = \beta_{i,j} - g_j$. Then $\gamma_{i,j}$ also satisfies the four properties of the claim. Moreover $\gamma_{i,j}^p - \gamma_{i,j} = c_j y_i^{-1} + d_j$. Hence replacing $\beta_{i,j}$ by $\gamma_{i,j}$, we may assume

$$(2.1) \quad \beta_{i,j}^p - \beta_{i,j} = c_j y_i^{-1} + d_j$$

Now for any j such that $i+1 < j \leq l$. We define $\beta_{i+1,j} = \beta_{i,j} - a_j \beta_{i,i+1}$ where $a_j \in k$ is such that $a_j^p = c_{i+1}^{-1} c_j$. Note that k is perfect so such an a_j exists.

We shall verify that these $\beta_{i+1,j}$ satisfy the four assertions of the claim. Firstly, since $L_{i+1} = L_i(\beta_{i,i+1})$, for $i+1 < n \leq l-1$ we have

$$L_{i+1}(\beta_{i+1,j}; i+1 < j \leq n) = L_i(\beta_{i,j}; i < j \leq n) = L_n$$

Hence the third property is satisfied.

We Compute

$$\begin{aligned} \beta_{i+1,j}^p - \beta_{i+1,j} &= \beta_{i,j}^p - \beta_{i,j} - a_j^p \beta_{i,i+1}^p + a_j \beta_{i,i+1} \\ &= c_j y_i^{-1} + d_j - a_j^p (\beta_{i,i+1} + c_{i+1} y_i^{-1} + d_{i+1}) + a_j \beta_{i,i+1} \\ &= (c_j - a_j^p c_{i+1}) y_i^{-1} + d_j - a_j^p d_{i+1} + (a_j - a_j^p) \beta_{i,i+1} \\ &= (a_j - a_j^p) \beta_{i,i+1} + d_j - a_j^p d_{i+1} \end{aligned}$$

Hence $\beta_{i+1,j}^p - \beta_{i+1,j} \in L_{i+1}$. If $a_j = a_j^p$ then $\beta_{i+1,j}^p - \beta_{i+1,j} \in k$ but this will lead to a residue field extension for S/R which contradicts the assumption that S/R is totally ramified. Hence $a_j \neq a_j^p$ and

$$(2.2) \quad \beta_{i+1,j}^p - \beta_{i+1,j} = (\text{nonzero constant}) \beta_{i,i+1} + \text{constant}$$

So $v_{i+1}(\beta_{i+1,j}^p - \beta_{i+1,j}) = v_{i+1}(\beta_{i,i+1}) = -1$. We have now verified the first two properties of the claim too.

Finally, $v_{i+2}(\beta_{i+1,i+2}^p) = v_{i+2}(\beta_{i+1,i+2}^p - \beta_{i+1,i+2}) = v_{i+2}(\beta_{i,i+1})$. So we deduce that $v_{i+2}(\beta_{i+1,i+2}) = p^{-1}v_{i+2}(\beta_{i,i+1}) = p^{-1}pv_{i+1}(\beta_{i,i+1}) = -1$. This completes the proof of the claim. \square

The field extension $L_{l-1}/\mathbf{QF}(R)$ is Galois with Galois group $(\mathbb{Z}/p\mathbb{Z})^{l-1}$ and $\text{Gal}(\mathbf{QF}(S)/L_{l-1}) = \mathbb{Z}/p\mathbb{Z}$. Moreover, both T_{l-1}/R and S/T_{l-1} are totally ramified extension. Note that $L_{l-1} = \mathbf{QF}(R)(\alpha_1, \dots, \alpha_{l-1})$. So by induction hypothesis $d_{T_{l-1}/R} = 2p^{l-1} - 2$.

Since $\mathbf{QF}(S) = L_{l-1}(\beta_{l-1,l})$, $\beta_{l-1,l}^p - \beta_{l-1,l} \in L_{l-1}$ and $v_{l-1}(\beta_{l-1,l}^p - \beta_{l-1,l}) = -1$, we have $d_{S/T_{l-1}} = 2p - 2$ by “ $l = 1$ case”.

Finally using the transitivity of different, we see that $d_{S/R} = e_{S/T_{l-1}}d_{T_{l-1}/R} + d_{S/T_{l-1}} = p(2p^{l-1} - 2) + 2p - 2 = 2p^l - 2$. This completes the proof of the lemma. \square

Proposition 2.5. *Let $i \geq 1$ and S/R be a finite extension of complete DVRs over a perfect field k of characteristic p such that $\text{Gal}(\mathbf{QF}(S)/\mathbf{QF}(R)) = G = G_i$. Let L be the subfield of $\mathbf{QF}(S)$ generated over $\mathbf{QF}(R)$ by all $\alpha \in \mathbf{QF}(S)$ such that $v_R(\alpha^p - \alpha) = -i$. Then $G_{i+1} \supset \text{Gal}(\mathbf{QF}(S)/L)$.*

Proof. Let $L' = \mathbf{QF}(S)^{G_{i+1}}$ and $H = \text{Gal}(\mathbf{QF}(S)/L) \leq G$. Let T and T' be the normalization of R in L and L' respectively. Since G_{i+1} is a normal subgroup of G , the extension $L'/\mathbf{QF}(R)$ is Galois and $\text{Gal}(L'/\mathbf{QF}(R)) = G/G_{i+1} (= \bar{G} \text{ say})$. Moreover the ramification filtration on \bar{G} is given by $\bar{G}_i = \bar{G}$ and $\bar{G}_{i+1} = \{e\}$ (Proposition 2.1). If $G_{i+1} = G$ then $H \subset G_{i+1}$ and we are done. So we may assume $G_{i+1} \neq G$. By Proposition 2.2 $\bar{G} \neq \{e\}$ is isomorphic to the direct sum of copies of $\mathbb{Z}/p\mathbb{Z}$.

Let $L'' \subset L'$ be any $\mathbb{Z}/p\mathbb{Z}$ -extension of $\mathbf{QF}(R)$. By Artin-Schrier theory there exists $\alpha \in L'' \setminus \mathbf{QF}(R)$ such that $\beta := \alpha^p - \alpha \in \mathbf{QF}(R)$. Let x be a local parameter of R then $R = k[[x]]$. If $v_R(\beta) > 0$ then $\alpha = c - \beta - \beta^p - \beta^{p^2} - \dots \in R$ for some $c \in \mathbb{F}_p$. So $v_R(\beta) \leq 0$. Moreover since $G_0 = G$, S/R is totally ramified. So $v_R(\beta) \neq 0$ and hence $v_R(\beta) \leq 0$. If $v_R(\beta)$ is a multiple of p then $\beta = c_0x^{pl} + c_1x^{pl+1} + \dots$, for some integer $l < 0$. Let $c \in k$ be such that $c^p = c_0$ and let $\alpha' = \alpha - cx^l$. Then $\beta' := \alpha'^p - \alpha' = \beta - c_0x^{pl} + cx^l$, $v_R(\beta') > v_R(\beta)$ and $L'' = \mathbf{QF}(R)(\alpha) = \mathbf{QF}(R)(\alpha')$. Hence by such modifications we may assume $v_R(\alpha^p - \alpha) = -r < 0$ is coprime to p . Let T'' be the normalization of R in L'' . By explicit calculation of the different and using Hilbert's different formula, the degree of the different $d_{T''/R} = (r+1)(p-1)$. Since \bar{G}_{i+1} is trivial and $\bar{G}_i = \bar{G}$, by Hilbert's different formula $d_{T'/R} = (i+1)|\bar{G}| - i - 1$. Let \bar{H} be the index p subgroup of \bar{G} such that $L'' = L^{\bar{H}}$. Then the ramification filtration on \bar{H} (coming from the extension T'/T'') is induced from \bar{G} . Hence $d_{T'/T''} = (i+1)|\bar{H}| - i - 1$. Using Lemma 2.3 and $e_{T'/T''} = |\bar{H}|$, we obtain

$$(i+1)|\bar{G}| - i - 1 = |\bar{H}|(r+1)(p-1) + (i+1)|\bar{H}| - i - 1$$

Using $|\bar{G}| = p|\bar{H}|$ above and solving for r , one gets $r = i$. Hence $L'' \subset L$. Since L'' was an arbitrary $\mathbb{Z}/p\mathbb{Z}$ -extension of $\mathbf{QF}(R)$ contained in L' and L' is generated by such $\mathbb{Z}/p\mathbb{Z}$ -extensions, we have that $L' \subset L$. So by the fundamental theorem of Galois theory $H \subset G_2$. \square

Proposition 2.6. *Let S/R be a finite extension of complete DVRs over a perfect field k of characteristic p such that $\text{Gal}(\text{QF}(S)/\text{QF}(R)) = G = G_1$. Let L be the subfield of $\text{QF}(S)$ generated over $\text{QF}(R)$ by all $\alpha \in \text{QF}(S)$ such that $v_R(\alpha^p - \alpha) = -1$. Then $G_2 = \text{Gal}(\text{QF}(S)/L)$.*

Proof. In view of Proposition 2.5, it is enough to show $G_2 \subset H := \text{Gal}(\text{QF}(S)/L)$. Let T be the normalization of R in L . Note that $L/\text{QF}(R)$ is a Galois extension with Galois group G/H . By Lemma 2.4 $d_{T/R} = 2|G/H| - 2$. So using Lemma 2.3 one gets:

$$2|G| - 2 + \sum_{i=2}^{\infty} (|G_i| - 1) = |H|(2|G/H| - 2) + 2|H| - 2 + \sum_{i=2}^{\infty} (|H_i| - 1)$$

Rearranging and using $|G| = |G/H| \cdot |H|$, the above reduces to the following

$$2|G/H| - 2 + |H|^{-1} \sum_{i=2}^{\infty} (|G_i| - |H_i|) = 2|G/H| - 2$$

So $G_i = H_i$ for $i \geq 2$. Hence $G_2 = H \cap G_2$ which implies $G_2 \subset H$. \square

Corollary 2.7. *Let S/R be a finite extension of complete DVRs over a perfect field k of characteristic p such that $\text{Gal}(\text{QF}(S)/\text{QF}(R)) = G = F^1G$. Then $F^2G \neq G$ iff there exists $\alpha \in \text{QF}(S)$ such that $\alpha^p - \alpha \in \text{QF}(R)$ and $v_R(\alpha^p - \alpha) = -1$.*

3. REDUCING INERTIA

For a local ring R , let m_R denote the maximal ideal of R . In this section we shall show how even the wild part of inertia subgroup of a Galois cover can be reduced. We begin with the following lemma.

Lemma 3.1. *Let R be a DVR and K be the quotient field of R . Let L and M be finite separable extensions of K and $\Omega = LM$ their compositum. Let A be a DVR dominating R with quotient field Ω . Note that $S = A \cap L$ and $T = A \cap M$ are DVRs. Let \hat{K} , \hat{L} , \hat{M} and $\hat{\Omega}$ be the quotient field of the complete DVRs \hat{R} , \hat{S} , \hat{T} and \hat{A} respectively. If $A/m_A = S/m_S$ then $\hat{\Omega} = \hat{L}\hat{M}$. Here all fields are viewed as subfields of an algebraic closure of \hat{K} .*

Proof. Note that \hat{L} and \hat{M} are contained in $\hat{\Omega}$. So $\hat{L}\hat{M} \subset \hat{\Omega}$. Let π_A denote a uniformizing parameter of A . Then $\pi_A \in LM \subset \hat{L}\hat{M}$. So it is enough to show that $\hat{\Omega} = \hat{L}[\pi_A]$. Note that $\hat{S}[\pi_A]$ is a finite \hat{S} -module, hence it is a complete DVR [Coh]. Also $\hat{S} \subset \hat{S}[\pi_A] \subset \hat{A}$ and π_A generate the maximal ideal of \hat{A} , hence $\pi_A S$ is the maximal ideal of $\hat{S}[\pi_A]$. Moreover, the residue field of \hat{S} is equal to $S/m_S = A/m_A$ which is same as the residue field of \hat{A} . Hence the residue field of $\hat{S}[\pi_A]$ is also same as the residue field of \hat{A} . So $\hat{S}[\pi_A] = \hat{A}$ (by [Coh, Lemma 4]). Hence the quotient field of $\hat{S}[\pi_A]$ is $\hat{\Omega}$. But that means $\hat{L}[\pi_A] = \hat{\Omega}$. \square

Corollary 3.2. *Let the notation be as in the above theorem. If $\hat{L} \subset \hat{M}$ then A/T is an unramified extension.*

Proof. Since Ω/M is finite extension, so is $\hat{\Omega}/\hat{M}$. Hence \hat{A} is a finite \hat{T} -module. By the above lemma and the hypothesis $\hat{\Omega} = \hat{M}$. So $\hat{A} = \hat{T}$, i.e. A/T is unramified. \square

Let k be any field.

Theorem 3.3. *Let $X \rightarrow Y$ and $Z \rightarrow Y$ be Galois covers of regular k -curves branched at $\tau \in Y$. Let τ_x and τ_z be closed points of X and Z respectively, lying above τ . Suppose $k(\tau_z) = k(\tau)$. Let W be an irreducible dominating component of the normalization of $X \times_Y Z$ containing the closed point (τ_x, τ_z) . Then $W \rightarrow Y$ is a Galois cover ramified at τ and the decomposition subgroup of the cover at τ is the Galois group of the field extension $QF(\hat{\mathcal{O}}_{X, \tau_x})QF(\hat{\mathcal{O}}_{Z, \tau_z})/QF(\hat{\mathcal{O}}_{Y, \tau})$.*

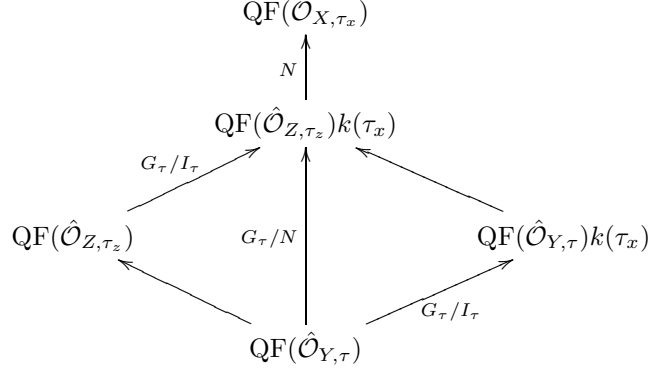
Proof. Let $R = \mathcal{O}_{Y, \tau}$. Note that R is a DVR. Let K be the quotient field of R . Let L and M be the function field of X and Z respectively and $\Omega = LM$ be their compositum. By definition W is an irreducible regular curve with function field Ω and the two projections give the covering morphisms to X and Y . Let τ_w denote the closed point $(\tau_x, \tau_z) \in W$ and $A = \mathcal{O}_{W, \tau_w}$. Since τ_w lies above τ_x under the covering $W \rightarrow X$ and above τ_z under the covering $W \rightarrow Z$, we have that $A \cap L = \mathcal{O}_{X, \tau_x} (= S \text{ say})$ and $A \cap M = \mathcal{O}_{Z, \tau_z} (= T \text{ say})$. Since $k(\tau_z) = k(\tau)$ and $k(W) = k(X)k(Z)$ we get that $k(\tau_w) = k(\tau_z)k(\tau_x) = k(\tau_x)$. But this is same as $A/m_A = S/m_S$. So using the above lemma, we conclude that $\hat{L}\hat{M} = \hat{\Omega}$.

The decomposition group of the cover $W \rightarrow Y$ at τ_w is given by the Galois group of the field extension $\hat{\Omega}/\hat{K}$ ([Bou, Corollary 4, Section 8.6, Chapter 6]). This completes the proof because $\hat{\Omega} = \hat{L}\hat{M} = QF(\hat{\mathcal{O}}_{X, \tau_x})QF(\hat{\mathcal{O}}_{Z, \tau_z})$ and $\hat{K} = QF(\hat{\mathcal{O}}_{Y, \tau})$. \square

Proposition 3.4. *Let $\Phi : X \rightarrow Y$ be a G -cover of regular k -curves ramified at $\tau_x \in X$ and let $\tau = \Phi(\tau_x)$. Let G_τ and I_τ be the decomposition subgroup and the inertia subgroup respectively at τ_x . Let $N \leq I_\tau$ be a normal subgroup of G_τ . Suppose there exist a Galois cover $\Psi : Z \rightarrow Y$ of regular k -curves ramified at $\tau_z \in Z$ with $\Psi(\tau_z) = \tau$ such that $k(\tau_z) = k(\tau)$ and the fixed field $QF(\hat{\mathcal{O}}_{X, \tau_x})^N$ is same as the compositum $QF(\hat{\mathcal{O}}_{Z, \tau_z})k(\tau_x)$. Let W be an irreducible dominating component of the normalization of $X \times_Y Z$ containing (τ_x, τ_z) . Then the natural morphism $W \rightarrow Z$ is a Galois cover. The inertia group and the decomposition group at the point (τ_x, τ_z) are N and an extension of N by $\text{Gal}(k(\tau_x)/k(\tau))$ respectively.*

Proof. Let $\tau_w \in W$ be the point (τ_x, τ_z) . Applying Theorem 3.3, we obtain that the decomposition group of the Galois cover $W \rightarrow Y$ at τ_w is isomorphic to $G_{\tau_w} = \text{Gal}(QF(\hat{\mathcal{O}}_{X, \tau_x})QF(\hat{\mathcal{O}}_{Z, \tau_z})/QF(\hat{\mathcal{O}}_{Y, \tau}))$. Since $QF(\hat{\mathcal{O}}_{Z, \tau_z}) \subset QF(\hat{\mathcal{O}}_{X, \tau_x})$, we have $G_{\tau_w} = G_\tau = \text{Gal}(QF(\hat{\mathcal{O}}_{X, \tau_x})/QF(\hat{\mathcal{O}}_{Y, \tau}))$. Since $k(\tau_z) = k(\tau)$, the inertia group and the decomposition group of the cover $Z \rightarrow Y$ at τ_z are both $\text{Gal}(QF(\hat{\mathcal{O}}_{Z, \tau_z})/QF(\hat{\mathcal{O}}_{Y, \tau}))$. Since $QF(\hat{\mathcal{O}}_{X, \tau_x})^N = QF(\hat{\mathcal{O}}_{Z, \tau_z})k(\tau_x)$ we also obtain that $\text{Gal}(QF(\hat{\mathcal{O}}_{Z, \tau_z})k(\tau_x)/QF(\hat{\mathcal{O}}_{Y, \tau})) = G_\tau/N$. Moreover, we have $G_\tau/I_\tau = \text{Gal}(k(\tau_x)/k(\tau)) = \text{Gal}(k(\tau_x)QF(\hat{\mathcal{O}}_{Y, \tau})/QF(\hat{\mathcal{O}}_{Y, \tau}))$. Since $\hat{\mathcal{O}}_{Z, \tau_z}/\hat{\mathcal{O}}_{Y, \tau}$ is totally

ramified, $\mathrm{QF}(\hat{\mathcal{O}}_{Z,\tau_z})$, $k(\tau_x)\mathrm{QF}(\hat{\mathcal{O}}_{Y,\tau})$ are linearly disjoint over $\mathrm{QF}(\hat{\mathcal{O}}_{Y,\tau})$.



So $\mathrm{Gal}(\mathrm{QF}(\hat{\mathcal{O}}_{Z,\tau_z})k(\tau_x)/\mathrm{QF}(\hat{\mathcal{O}}_{Z,\tau_z})) = \mathrm{Gal}(k(\tau_x)/k(\tau))$. So the decomposition group of $W \rightarrow Z$ is $\mathrm{Gal}(\mathrm{QF}(\hat{\mathcal{O}}_{X,\tau})/\mathrm{QF}(\hat{\mathcal{O}}_{Z,\tau_z}))$ which is an extension of N by $\mathrm{Gal}(k(\tau_x)/k(\tau))$ and the inertia group is $\mathrm{Gal}(\mathrm{QF}(\hat{\mathcal{O}}_{X,\tau})/\mathrm{QF}(\hat{\mathcal{O}}_{Z,\tau_z})k(\tau_x)) = N$. \square

Let k be an algebraically closed field of characteristic $p > 0$.

Theorem 3.5. *Let $\Phi : X \rightarrow Y$ be a G -Galois cover of regular k -curves. Let $\tau_x \in X$ be a ramification point and $\tau = \Phi(\tau_x)$. Let I be the inertia group of Φ at τ_x . There exists a cover $\Psi : Z \rightarrow Y$ of deg $|I|$, such that the cover $W \rightarrow Z$ is étale over τ_z where W is the normalization of $X \times_Y Z$ and $\tau_z \in Z$ is such that $\Psi(\tau_z) = \tau$. Moreover if there are no non-trivial homomorphism from $G \rightarrow P$ where P is a p -sylog subgroup of I then $W \rightarrow Z$ is a G -cover of irreducible regular k -curves.*

Proof. Since I is the inertia group, it is isomorphic to $P \rtimes \mu_n$ where $(p, n) = 1$ and μ_n is a cyclic group of order n . Let y be a local coordinate of Y at τ such that $k(Y)[y^{1/n}] \cap k(X) = k(Y)$. Let Z_1 be the normalization of Y in $k(Y)[y^{1/n}]$. Then $Z_1 \rightarrow Y$ is a μ_n -cover branched at τ such that $k(Z_1)$ and $k(X)$ are linearly disjoint over $k(Y)$. Let $\tau_{z1} \in Z_1$ be a point lying above τ . Let X_1 be the normalization of $X \times_Y Z_1$. Then by the above theorem $\Phi_1 : X_1 \rightarrow Z_1$ is a G -cover of irreducible regular k -curves and the inertia group at (τ_x, τ_{z1}) is P .

Let $Y_1 = Z_1$, $\tau_{x1} = (\tau_x, \tau_{z1})$ and $\tau_1 = \tau_{z1}$. Then $\Phi_1 : X_1 \rightarrow Y_1$ is a G -cover with $\Phi_1(\tau_{x1}) = \tau_1$ and the inertia group of this cover at τ_{x1} is P . Let y_1 be a regular parameter of Y_1 at τ_1 . Then $k(Y_1)/k(y_1)$ is a finite extension. Since Y_1 is a regular curve, we get a finite morphism $\alpha : Y_1 \rightarrow \mathbb{P}_{y_1}^1$ such that $\alpha(\tau_1)$ is the point $y_1 = 0$ and α is étale at τ_1 (as $\hat{\mathcal{O}}_{Y_1,\tau_1} = k[[y_1]]$).

Note that $\mathrm{QF}(\hat{\mathcal{O}}_{X,\tau_{x1}})/k((y_1))$ is a P -extension. By [Ha, Cor 2.4], there exist a P -cover $V \rightarrow \mathbb{P}_{y_1}^1$ branched only at $y_1 = 0$ (where it is totally ramified) such that $\mathrm{QF}(\hat{\mathcal{O}}_{V,\theta}) = \mathrm{QF}(\hat{\mathcal{O}}_{X_1,\tau_{x1}})$ as extensions of $k((y_1))$. Here θ is the unique point in V lying above $y_1 = 0$. Since $V \rightarrow \mathbb{P}_{y_1}^1$ is totally ramified over $y_1 = 0$ and $Y_1 \rightarrow \mathbb{P}_{y_1}^1$ is étale over $y_1 = 0$, the two covers are linearly disjoint. Let Z be the normalization of $V \times_{\mathbb{P}_{y_1}^1} Y_1$. Then the projection map $Z \rightarrow Y_1$ is a P -cover. Let $\tau_z \in Z$ be the closed point (θ, τ_1) . By Lemma 3.1, $\mathrm{QF}(\hat{\mathcal{O}}_{Z,\tau_z}) = \mathrm{QF}(\hat{\mathcal{O}}_{V,\theta})\mathrm{QF}(\hat{\mathcal{O}}_{Y_1,\tau_1}) = \mathrm{QF}(\hat{\mathcal{O}}_{X_1,\tau_{x1}})$. Applying Proposition 3.4 with $N = \{e\}$, we get that an irreducible dominating component W of the normalization of $X_1 \times_{Y_1} Z$ is a Galois cover of Z such that

the inertia group over τ_z is $\{e\}$. Hence the normalization of $X_1 \times_{Y_1} Z$ is a cover of Z étale over τ_z .

Moreover, there are no nontrivial homomorphism from G to P implies that $k(Z)$ and $k(X_1)$ are linearly disjoint over $k(Y_1)$. Hence $W \rightarrow Z$ is a G -cover. We take $Z \rightarrow Y$ to be the composition $Z \rightarrow Y_1 \rightarrow Y$. Note that the morphism $X \times_Y Z \rightarrow Z$ is same as $X_1 \times_{Y_1} Z \rightarrow Z$ and the degree of the morphism $Z \rightarrow Y$ is $|P|n = |I|$. \square

Theorem 3.6. *Let $\Phi : X \rightarrow \mathbb{P}^1$ be a G -Galois cover of regular k -curves. Suppose Φ is branched only at one point $\infty \in \mathbb{P}^1$ and the inertia group of Φ over ∞ is I . Let P be a subgroup of I such that $I_1 \supset P \supset I_2$. Suppose there are no nontrivial homomorphism from G to P . Then there exist a G -cover $W \rightarrow \mathbb{P}^1$ ramified only at ∞ and the inertia group at ∞ is P .*

Proof. Let $n = [I : I_1]$ be the tame ramification index of Φ at ∞ . Let x be a local coordinate on \mathbb{P}^1 and the point ∞ is $x = \infty$. Let $\mathbb{P}_y^1 \rightarrow \mathbb{P}_x^1$ be the Kummer cover obtained by sending y^n to x . Since Φ is étale at $x = 0$ and the cover $\mathbb{P}_y^1 \rightarrow \mathbb{P}_x^1$ is totally ramified at $x = 0$ the two covers are linearly disjoint. So letting W to be the normalization of $X \times_{\mathbb{P}_x^1} \mathbb{P}_y^1$, we obtain a G -cover $\Phi_1 : W \rightarrow \mathbb{P}_y^1$ of regular k -curves. Moreover by Abhyankar's lemma Φ_1 is ramified only at $y = \infty$ and the inertia group of Φ_1 at $y = \infty$ is same the subgroup I_1 of I . So replacing Φ by Φ_1 , we may assume $I = I_1$. Also since I_1/I_2 is abelian, P is a normal subgroup of I .

Let $\tau \in X$ be a point above $x = \infty$. Let $S = \hat{\mathcal{O}}_{X,\tau}$ and $R = \hat{\mathcal{O}}_{\mathbb{P}^1,\infty}$ then $R = k[[x^{-1}]]$ and $\text{Gal}(QF(S)/QF(R)) = I$. Let $L = QF(S)^P$. Then by Proposition 2.6, $L = QF(R)(\alpha_1, \dots, \alpha_l)$ where $\alpha_i \in QF(S)$ is such that $v_R(\alpha_i^P - \alpha_i) = -1$ for $1 \leq i \leq l$. Let T be the normalization of R in L . Then $\text{Spec}(T)$ is a principal P -cover of $\text{Spec}(R)$. By [Ha, Corollary 2.4], this extends to a P -cover $\Psi : Z \rightarrow \mathbb{P}_x^1$ ramified only at $x = \infty$ where it is totally ramified. Let $\tau_z \in Z$ be the point lying above $x = \infty$ then $QF(\hat{\mathcal{O}}_{Z,\tau_z}) = L = QF(S)^P$. By Lemma 2.4 $d_{T/R} = 2|P| - 2$. So by Riemann-Hurwitz formula, the genus of Z is given by

$$2g_Z - 2 = |P|(0 - 2) + d_{T/R}$$

Hence $g_Z = 0$. So Z is isomorphic to \mathbb{P}^1 .

Since there are no nontrivial homomorphism from G to P , Φ and Ψ are linearly disjoint covers of \mathbb{P}_x^1 . Let W be the normalization of $X \times_{\mathbb{P}_x^1} Z$. Now we are in the situation of Proposition 3.4. Hence the G -cover $W \rightarrow Z$ is ramified only at τ_z and the inertia group at τ_z is P . This completes the proof as Z is isomorphic to \mathbb{P}^1 . \square

Remark 3.7. Note that if G is a simple group different from $\mathbb{Z}/p\mathbb{Z}$ then there are no nontrivial homomorphism from G to P . Hence the above results apply in this scenario.

Corollary 3.8. *Let $\Phi : X \rightarrow \mathbb{P}^1$ be a G -Galois cover of regular k -curves branched only at one point $\infty \in \mathbb{P}^1$ and the inertia group of Φ over ∞ is I . Suppose there are no nontrivial homomorphism from G to I_2 . Then conjugates of I_2 generate G .*

Proof. Applying the above theorem with $P = I_2$, we get an étale G -cover of \mathbb{A}^1 with the inertia group I_2 at ∞ . Hence the conjugates of I_2 generate G since a nontrivial étale cover of \mathbb{A}^1 must be wildly ramified over ∞ . \square

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